SPECTRAL THEORY OF SYMMETRIC ORDINARY DIFFERENTIAL OPERATORS WITH AN INDEFINITE WEIGHT FUNCTION

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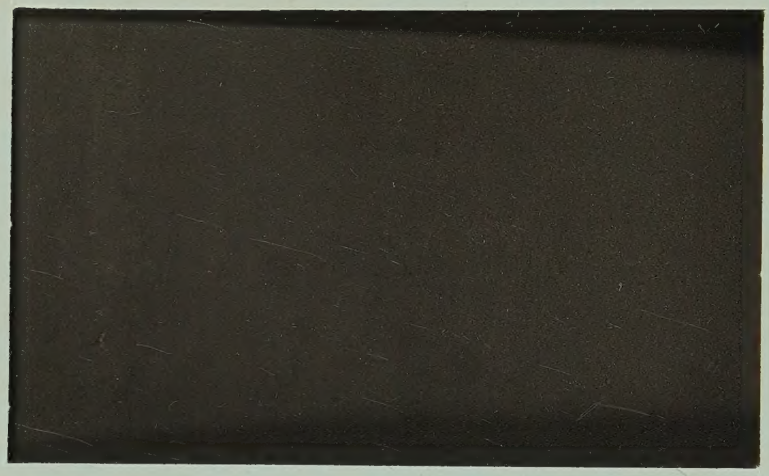

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4. THE KREIN TRANSFORMATION AND PARSEVAL RELATION IN THE INDEFINITE CASE.

4.1. The Krein transformation.

Let H be a linear space and T a linear operator in H with domain $D(T) \subset H$. For each complex λ, let a system of linear functionals $\Phi_j(f;\lambda)$ [1] $(f \in H,\ j=1,\dots,n)$ be defined on H. This system of linear functionals is called a Krein transformation corresponding to the operator T (a system of a directing functionals in the terminology of M.G. Krein [17], [18], see also M.G. Krein, H. Langer [20]), if the following conditions are satisfied:

(i) For $f \in H$ fixed, $\Phi_j(f;\lambda)$ is an entire function of λ.

(ii) For at least one value of λ the functionals $\Phi_j(f;\lambda)$ $(f \in H,\ j=1,\dots,n)$ are linearly independent.

(iii) For $f \in H$ and $\lambda_0 \in \mathbb{C}$ the equation $Tu - \lambda_0 u = f$ has a solution $u \in D(T)$ if and only if $\Phi_j(f,\lambda_0) = 0$, $j=1,\dots,n$.

Assume that $\{\Phi_1,\dots,\Phi_n\}$ is a Krein transformation corresponding to the operator T in H. As a general property of the Krein transformation we have the following

LEMMA 4.1.1. If $g \in D(T)$, then for $\lambda \in \mathbb{C}$

$$\Phi_j(Tg;\lambda) = \lambda \Phi_j(g;\lambda), \quad j=1,\dots,n.$$

The following Lemmas 4.1.2, 4.1.3 and Proposition 4.1.4 are taken from M.G. Krein and H. Langer [20].

LEMMA 4.1.2. Let λ_0 be real and let n_0 be a non-negative integer. Suppose there exists a system of elements $v_1,\dots,v_n$ belonging to $D(T^{n_0})$ with property

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1) For the special operators considered in the following the Krein transformation coincides with the system $\Phi_j(\cdot;\lambda)$ introduced in (3.4.5), see Section 4.3.

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$$\det\,(\Phi_j(v_k;\lambda_0))_{j,k=1}^{n} \neq 0\,.$$

Then there exists a system of elements $u_1,\dots,u_n$, belonging to $D(T^{n_0})$, such that

$$\Phi_j(u_k;\lambda_0) = \delta_{jk}\,, \quad j,k=1,2,\dots,n\,.$$

In the following we write

$$\Phi_j^{(\mu)}(g;\lambda_0) = \frac{\partial^\mu \Phi_j(g;\lambda)}{\partial\lambda^\mu}\Big|_{\lambda=\lambda_0}\,.$$

LEMMA 4.1.3. Under the assumptions of Lemma 4.1.2 there exist elements $f_j^{(0)}, f_j^{(1)},\dots,f_j^{(n_0)}$, $j=1,\dots,n$, such that

$$\Phi_k^{(\mu)}(f_j^{(\nu)};\lambda_0) = \delta_{kj}\delta_{\mu\nu}\,, \quad j,k=1,2,\dots,n;\ \mu,\nu=0,1,\dots,n_0\,.$$

Proof. By Lemma 4.1.2, there exist elements $u_1,\dots,u_n \in D(T^{n_0})$ such that $\Phi_j(u_k;\lambda_0) = \delta_{jk}$, $j,k=1,2,\dots,n$. Define

$$\tilde{f}_j^{(\nu)} = \frac{(T-\lambda_0 I)^\nu}{\nu!}\,u_j\,, \quad j=1,\dots,n\,,\ \nu=0,1,\dots,n_0\,.$$

Then, by Lemma 4.1.1, we have

$$\Phi_j(\tilde{f}_j^{(\nu)};\lambda) = \Phi_j\Big(\frac{(T-\lambda_0 I)^\nu}{\nu!}\,u_j;\lambda\Big) = \frac{(\lambda-\lambda_0)^\nu}{\nu!}\,\Phi_j(u_j;\lambda)$$

and

$$\Phi_j^{(\mu)}(\tilde{f}_j^{(\nu)};\lambda) = \delta_{\mu\nu}\,, \quad \mu \le \nu,\ \nu=0,1,\dots,n_0\,.$$

Put

$$\tilde{\tilde{f}}_j^{(\nu)} = \sum_{s=\nu}^{n_0} c_s\,\tilde{f}_j^{(s)}\,, \text{ with some constants } c_s\,.$$

Then

$$\Phi_j^{(\mu)}(\tilde{\tilde{f}}_j^{(\nu)};\lambda_0) = \Phi_j^{(\mu)}\Big(\sum_{s=\nu}^{n_0} c_s\,\tilde{f}_j^{(s)};\lambda_0\Big) = \sum_{s=\nu}^{n_0} c_s\,\Phi_j^{(\mu)}(\tilde{f}_j^{(s)};\lambda_0)\,,$$

and it is easy to see that the c_s can be chosen in such a way that

$$\Phi_j^{(\mu)}(\tilde{\tilde{f}}_j^{(\mu)};\lambda_0) = \delta_{\mu\nu}\,, \quad \mu,\nu=0,1,\dots,n_0\,. \tag{4.1.1}$$

The elements $\tilde{f}_i^{(\nu)}$ have for $i \neq j$ the properties

$$\Phi_j^{(\mu)}(\tilde{\tilde{f}}_i^{(\nu)};\lambda_0) = 0, \quad \mu \le \nu, \ \nu = 0,1,\dots,n_0. \tag{4.1.2}$$

From (4.1.1) and (4.1.2) it follows

$$\Phi_j^{(\mu)}(\tilde{\tilde{f}}_i^{(\nu)};\lambda_0) = 0, \quad i \neq j, \ \mu \le \nu, \ \nu = 0,1,\dots,n_0. \tag{4.1.3}$$

We consider the matrix

$$Q = \begin{pmatrix} \Phi_1^{(\mu)}(\tilde{\tilde{f}}_1^{(\nu)};\lambda_0) & \Phi_2^{(\mu)}(\tilde{\tilde{f}}_1^{(\nu)};\lambda_0) & \cdots & \Phi_n^{(\mu)}(\tilde{\tilde{f}}_1^{(\nu)};\lambda_0) \\ \Phi_1^{(\mu)}(\tilde{\tilde{f}}_2^{(\nu)};\lambda_0) & \Phi_2^{(\mu)}(\tilde{\tilde{f}}_2^{(\nu)};\lambda_0) & \cdots & \Phi_n^{(\mu)}(\tilde{\tilde{f}}_2^{(\nu)};\lambda_0) \\ \vdots & & & \\ \Phi_1^{(\mu)}(\tilde{\tilde{f}}_n^{(\nu)};\lambda_0) & \Phi_2^{(\mu)}(\tilde{\tilde{f}}_n^{(\nu)};\lambda_0) & \cdots & \Phi_n^{(\mu)}(\tilde{\tilde{f}}_n^{(\nu)};\lambda_0) \end{pmatrix}$$

where every element is an $(n_0 \times n_0)$-matrix. From (4.1.1) the matrices on the mean diagonal of Q, i.e. $\Phi_1^{(\mu)}(\tilde{\tilde{f}}_1;\lambda_0),\dots, \Phi_n^{(\mu)}(\tilde{\tilde{f}}_n^{(\nu)};\lambda_0)$ are unit matrices and from (4.1.3) the other matrices are triangular matrices with zeros on and below the main diagonal. It follows that $\det Q = 1$, hence there exists a system of elements $f_j^{(\nu)}$ $(j = 1,\dots,n, \ \nu = 0,1,\dots,n_0)$ with the stated properties.

Now, let mutually different real numbers $\lambda_1,\dots,\lambda_\gamma$ and integers $\tilde{n}_1, \tilde{n}_2,\dots,\tilde{n}_\gamma$ be given. Moreover, assume that there exist elements $v_1,\dots,v_n \in D(T^N)$,

$$N \ge \sum_{s=1}^{\gamma} \tilde{n}_s + \gamma, \text{ with the property}$$

$$\det(\Phi_j(v_k;\lambda))_{j,k=1}^{n} \not\equiv 0.$$

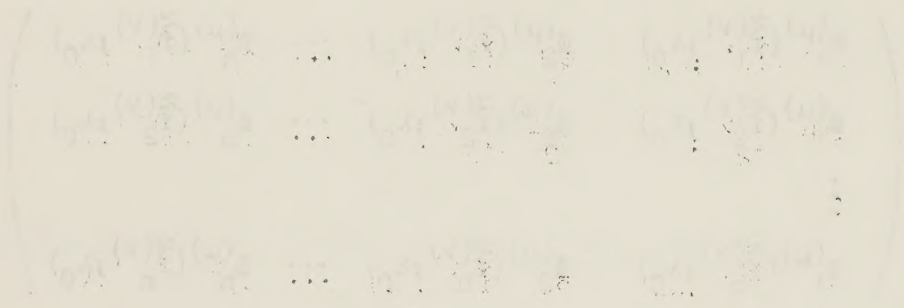
Then, by Lemma 4.1.2, to each λ_s we can find elements $u_{1,s},\dots,u_{n,s}$ belonging to $D(T^N)$ such that

$$\Phi_j(u_{k,s};\lambda_s) = \delta_{jk}, \quad j,k = 1,\dots,n, \ s = 1,\dots,\gamma. \tag{4.1.4}$$

Define

$$\tilde{u}_{k,s} = \left(\prod_{\substack{t=1 \\ t \neq s}}^{\gamma} \frac{(T - \lambda_k)^{\tilde{n}_t+1}}{\lambda_s - \lambda_k} \right) u_{k,s}.$$

Then (4.1.4) holds for $\tilde{u}_{k,s}$ instead of $u_{k,s}$ and we have



$$\tilde{u}_{k,s} \in D(T^{N_1}), \quad N_1 = N - \sum_{t=1}^{\gamma} \tilde{n}_t - \gamma + \tilde{n}_s + 1, \quad k = 1,\dots,n; \; s = 1,\dots,\gamma.$$

Now, using the construction in the proof of Lemma 4.1.3 for $n_0 = \tilde{n}_s$, $\lambda_0 = \lambda_s$ and $u_j = u_{j,s}$ $(j = 1,\dots,n)$, we get some system $\{f_{k,s}^{(\nu)}\}$ with the property

$$\Phi_j^{(\mu)}(f_{k,s}^{(\nu)};\lambda_s) = \delta_{jk}\,\delta_{\mu\nu}, \quad f_{k,s}^{(\nu)} \in D(T^{N_2}),$$

where $N_2 = N - \Sigma_{s=1}^{\gamma} \tilde{n}_s - \gamma + 1 \geq 1$.

Moreover, for $t,s = 1,2,\dots,\gamma$; $t \neq s$, we get

$$\Phi_j^{(\mu)}(\tilde{u}_{k,s};\lambda_t) = 0, \quad \mu = 0,1,\dots,\tilde{n}_s,$$

hence

$$\Phi_j^{(\mu)}(f_{k,s}^{(\nu)};\lambda_t) = 0, \quad \mu = 0,1,\dots,\tilde{n}_s.$$

We summarize this result as:

PROPOSITION 4.1.4. Let mutually different real numbers $\lambda_1, \lambda_2,\dots,\lambda_\gamma$ and non-negative integers $\tilde{n}_1,\dots,\tilde{n}_\gamma$ be given. Moreover, assume there exist elements $v_1,\dots,v_n \in D(T^N)$, where $N \geq \Sigma_{s=1}^{\gamma} \tilde{n}_s + \gamma$, such that

$$\det(\Phi_j(v_k,\lambda))_{j,k=1}^{n} \not\equiv 0.$$

Then, there exist elements $f_{k,s}^{(\nu)} \in D(T)$ with the property

$$\Phi_j^{(\mu)}(f_{k,s}^{(\nu)};\lambda_t) = \delta_{jk}\delta_{\mu\nu}\delta_{st}, \quad j,k = 1,2,\dots,n; \; \mu,\nu = 0,1,\dots,\tilde{n}_s; \; s,t = 0,1,\dots,\gamma.$$

4.2. Parseval relation.

In the following we suppose that on H an indefinite scalar product $[\cdot,\cdot]$ is given and that T is symmetric with respect to this scalar product. Assume $\{\Phi_1,\dots,\Phi_n\}$ is a Krein transformation corresponding to the operator T.

Let S_n be the set of all hermitian $(n\times n)$-matrix functions $\sigma(\lambda) = (\sigma_{ij}(\lambda))_{i,j=1}^{n}$, $-\infty < \lambda < \infty$, with the following properties:

(i) For arbitrary complex numbers $\xi_1,\dots,\xi_n$ the form

$$\sum_{i,j=1}^{n} \sigma_{ij}(\lambda)\xi_i\overline{\xi_j}$$

is a real and non-decreasing function of λ.

(ii) $\quad \sigma(0) = 0, \ \sigma(\lambda - 0) = \sigma(\lambda) \qquad (-\infty < \lambda < \infty)$.

From the property (i) it follows that on each bounded interval of the λ-axis, each function $\sigma_{ij}(\lambda)$ $(i,j = 1,\ldots,n)$ is of bounded variation and therefore the limits $\sigma_{ij}(\lambda - 0)$ exist for all real λ.

For a polynomial p of degree κ, let λ_s $(s = 0,1,\ldots,\gamma)$ be its mutually different zeros with multiplicities n_s $(s = 0,1,\ldots,\gamma)$.

The following theorem was stated by M.G. Krein and H. Langer in [20] without proof.

THEOREM 4.2.1. Let the following conditions be satisfied:

1) For a polynomial p of degree κ we have

$$[T\,p(T)\,u,\ p(T)\,u] \geq 0 \qquad (u \in D(T^{\kappa+1})).$$

2) There exist elements $v_1,\ldots,v_n$ belonging to $D(T^{\kappa+1})$ such that

$$\det(\Phi_j(v_k;\lambda))_{j,k=1}^{n} \not\equiv 0.$$

3) For arbitrary $u \in D(T^{\kappa+1})$ there exists a sequence $(u_n) \in D(T^{\kappa+2})$ such that

$$[T\,p(T)\,(u - u_n),\ p(T)\,(u - u_n)] \to 0 \qquad (n \to \infty).$$

Then there exists a function $\sigma \in S_n$, such that for $u \in H$ and $v \in D(T)$ we have

$$[u,v] = \sum_{i,j=1}^{n} \int_{-\infty}^{\infty} \Big(\Phi_i(u;\lambda) - \sum_{k=1}^{n}\sum_{s=0}^{\gamma}\sum_{\nu=0}^{\tilde{n}_s} \Phi_k^{(\nu)}(u;\lambda_s)\overline{\Phi_i(f_{k,s}^{(\nu)};\lambda)}\Big) \cdot$$

$$\cdot \Big(\Phi_j(v;\lambda) - \sum_{k=1}^{n}\sum_{s=0}^{\gamma}\sum_{\nu=0}^{n_s-1} \Phi_k^{(\nu)}(v;\lambda_s)\overline{\Phi_j(f_{k,s}^{(\nu)};\lambda)}\Big)\frac{d\sigma_{ij}(\lambda)}{\lambda|p(\lambda)|^2} +$$

$$+ \sum_{k=1}^{n}\sum_{s=0}^{\gamma}\sum_{\nu=0}^{\tilde{n}_s} \Phi_k^{(\nu)}(u;\lambda_s)\,[f_{k,s}^{(\nu)}, v] +$$

$$+ \sum_{k=1}^{n}\sum_{s=0}^{n}\sum_{\nu=0}^{n_s-1} \overline{\Phi_k^{(\nu)}(v;\lambda_s)}\,[u, f_{k,s}^{(\nu)}] -$$

$$- \sum_{k,m=1}^{n} \sum_{s,t=0}^{\gamma} \sum_{\nu=0}^{\tilde{n}_s} \sum_{\mu=0}^{n_s-1} \Phi_k^{(\nu)}(u;\lambda_s)\overline{\Phi_m^{(\mu)}(v;\lambda_t)} .$$

Here the elements $f_{k,s}^{(\nu)}$ are chosen such that $f_{k,s}^{(\nu)} \in D(T)$

$$\Phi_j^{(\mu)}(f_{k,s}^{(\nu)};\lambda_t) = \delta_{jk}\delta_{\mu\nu}\delta_{st}, \quad j,k=1,2,\ldots,n \tag{4.2.1}$$

$\mu,\nu = 0,1,\ldots,n_s-1$ if $s>0$, $\mu,\nu = 0,1,\ldots,n_0$ if $s=0$, $s,t = 0,1,\ldots,\gamma$, and $\tilde{n}_s = n_s - 1$ if $s>0$, $\tilde{n}_0 = n_0$.

For $x,y \in D(T^{K+1})$ we have

$$\left[T\,p(T)\,x,\ p(T)\,y\right] = \sum_{i,j=1}^{n} \int_{-\infty}^{\infty} \Phi_i(x;\lambda)\overline{\Phi(y;\lambda)}\,d\sigma_{ij}(\lambda) . \tag{4.2.2}$$

The integrals in Theorem 4.2.1 are to be understood as unproper integrales.

Proof. Set $L = D(T^{K+1})$ and introduce the possibly degenerated scalar product

$$\{x,y\} = \left[T\,p(T)\,x,\ p(T)\,y\right] \qquad (x,y \in L) .$$

The restriction of the operator T to $D(T^{K+2})$ defines a symmetric operator on L:

$$\{Tf,g\} = \{f,Tg\} \qquad (f,g \in D(T^{K+2})) .$$

The restriction of the Krein transformation to L is a Krein transformation corresponding to the restriction of T to $D(T^{K+2})$. To see this we have only to check the condition (iii), i.e. that for given f belonging to L and $\lambda \in \mathbb{C}$, the relation

$$\Phi_j(f;\lambda) = 0 \qquad j=1,\ldots,n \tag{4.2.3}$$

holds if and only if the equation

$$(T-\lambda I)\,g = f , \tag{4.2.4}$$

has a solution $g \in D(T^{K+2})$.

Indeed, it is obvious that (4.2.4) implies (4.2.3). On the other hand, $\Phi_j(f;\lambda) = 0$, $\overbrace{j=1,\ldots,n,}$ for some $f \in L = D(T^{K+1})$ implies that $(T-\lambda I)g = f$

has a solution $g \in D(T)$. From the equation $Tg = \lambda g + f$ it is easy to check by induction that g belongs even to $D(T^{K+2})$.

By a fundamental theorem of M.G. Krein [17], there exists a function $\sigma \in S_n$ such that for arbitrary elements x,y of L, the following representation holds true:

$$\{x,y\} = \sum_{i,j=1}^{n} \int_{-\infty}^{\infty} \Phi_i(x;\lambda)\, \overline{\Phi_j(y;\lambda)}\, d\sigma_{ij}(\lambda) . \tag{4.2.5}$$

For $u \in H$ and $v \in D(T)$ define

$$\tilde{u} = u - \sum_{k=0}^{n} \sum_{s=0}^{\gamma} \sum_{\nu=0}^{\tilde{n}_s} \Phi_k^{(\nu)}(u;\lambda_s) f_{k,s}^{(\nu)} , \tag{4.2.6}$$

where $\tilde{n}_s = n_s - 1$ $(s \neq 0)$, and $\tilde{n}_0 = n_0$,

$$\tilde{v} = v - \sum_{k=0}^{n} \sum_{s=0}^{\gamma} \sum_{\nu=0}^{n_s-1} \Phi_k^{(\nu)}(v;\lambda_s) f_{k,s}^{(\nu)} . \quad ^{1)} \tag{4.2.7}$$

Here the elements $f_{k,s}^{(\nu)}$ are taken from Proposition 4.1.3. Applying $\Phi_i(\cdot,\lambda)$ $(\Phi_j(\cdot,\lambda))$ to (4.2.6) ((4.2.7) resp.) gives

$$\Phi_i(\tilde{u};\lambda) = \Phi_i(u;\lambda) - \sum_{k=0}^{n} \sum_{s=0}^{\gamma} \sum_{\nu=0}^{\tilde{n}_s} \Phi_k^{(\nu)}(u;\lambda_s) \Phi_i(f_{k,s}^{(\nu)};\lambda) \tag{4.2.8}$$

$$\Phi_j(\tilde{v};\lambda) = \Phi_j(v;\lambda) - \sum_{k=0}^{n} \sum_{s=0}^{\gamma} \sum_{\nu=0}^{n_s-1} \Phi_k^{(\nu)}(v;\lambda_s) \Phi_j(f_{k,s}^{(\nu)};\lambda) \tag{4.2.9}$$

Taking the μ-th derivative in these relations at $\lambda = \lambda_t$ and using the properties of the elements $f_{ks}^{(\nu)}$ we get

$$\Phi_i^{(\mu)}(\tilde{u};\lambda_t) = \Phi_i^{(\mu)}(u;\lambda_t) - \sum_{k=0}^{n} \sum_{s=0}^{\gamma} \sum_{\nu=0}^{\tilde{n}_s} \Phi_k^{(\nu)}(u;\lambda_s) \delta_{ik}\delta_{\mu\nu}\delta_{st} ,$$

$$\Phi_j^{(\mu)}(\tilde{v};\lambda_t) = \Phi_j^{(\mu)}(v;\lambda_t) - \sum_{k=0}^{n} \sum_{s=0}^{\gamma} \sum_{\nu=0}^{n_s-1} \Phi_k^{(\nu)}(v;\lambda_s) \delta_{jk}\delta_{\mu\nu}\delta_{st} .$$

Hence, for all $t = 0,1,\ldots,\gamma$ and $i,j = 1,2,\ldots,n$

$$\Phi_i^{(\mu)}(\tilde{u};\lambda_t) = 0 , \quad \mu = 0,1,\ldots,\tilde{n}_t \tag{4.2.10}$$

$$\Phi_j^{(\mu)}(\tilde{v};\lambda_t) = 0 , \quad \mu = 0,1,\ldots,n_t - 1 . \tag{4.2.11}$$

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1) $\sum_{\nu=0}^{-1}$ is to be understood as zero.

We shall show that these relations imply the existence of elements $\tilde{\tilde{u}}, \tilde{\tilde{v}} \in L$ such that

$$\tilde{u} = Tp(T)\tilde{\tilde{u}}, \quad \tilde{v} = p(T)\tilde{\tilde{v}}. \tag{4.2.12}$$

Indeed, if $p(\lambda) = \prod_{t=0}^{\gamma} (\lambda - \lambda_t)^{n_t}$, $n_0 \geq 0$, $n_t \geq 1$, $t = 1, \ldots, \gamma$, then $\Phi_j(\tilde{u};\lambda_t) = 0$ (for some fixed t) implies the existence of an element $\tilde{u}_1$, such that $\tilde{u} = (T - \lambda_t I)\tilde{u}_1$. Then, if $n_t > 1$, we have $\Phi(\tilde{u}_1;\lambda) = \frac{1}{\lambda - \lambda_t}\Phi(\tilde{u};\lambda)$, which also vanishes at $\lambda = \lambda_t$. It follows that there exists an element $\tilde{u}_2$, such that $\tilde{u}_1 = (T - \lambda_t I)\tilde{u}_2$, or $\tilde{u} = (T - \lambda_t I)^2 \tilde{u}_2$. Continuing this procedure gives the relations (4.2.12).

From (4.2.5) we find

$$\begin{aligned}
[\tilde{u},\tilde{v}] &= [Tp(T)\tilde{\tilde{u}},\ p(T)\tilde{\tilde{v}}] \\
&= \{\tilde{\tilde{u}},\tilde{\tilde{v}}\} = \sum_{i,j=1}^{n} \int_{-\infty}^{\infty} \Phi_i(\tilde{\tilde{u}};\lambda)\overline{\Phi_j(\tilde{\tilde{v}};\lambda)}\, d\sigma_{ij}(\lambda) \\
&= \sum_{i,j=1}^{n} \int_{-\infty}^{\infty} \Phi_i(\tilde{u};\lambda)\overline{\Phi_j(\tilde{v};\lambda)}\, \frac{d\sigma_{ij}(\lambda)}{\lambda |p(\lambda)|^2}.
\end{aligned} \tag{4.2.13}$$

From (4.2.6) and (4.2.7) the left hand side of (4.2.13) can be written as

$$\begin{aligned}
[\tilde{u},\tilde{v}] &= \Big[u - \sum_{k=0}^{n}\sum_{s=0}^{\gamma}\sum_{\nu=0}^{\tilde{n}_s} \Phi_k^{(\nu)}(u;\lambda_s) f_{k,s}^{(\nu)},\ v - \sum_{k=0}^{n}\sum_{s=0}^{\gamma}\sum_{\nu=0}^{\tilde{n}_s - 1} \Phi_k^{(\nu)}(v;\lambda_s) f_{k,s}^{(\nu)}\Big] \\
&= [u,v] - \sum_{k=0}^{n}\sum_{s=0}^{\gamma}\sum_{\nu=0}^{\tilde{n}_s} \Phi_k^{(\nu)}(u;\lambda_s)\left[f_{k,s}^{(\nu)}, v\right] - \\
&- \sum_{k=0}^{n}\sum_{s=0}^{\gamma}\sum_{\nu=0}^{n_s - 1} \overline{\Phi_k^{(\nu)}(v;\lambda_s)}\ \left[u, f_{k,s}^{(\nu)}\right] + \\
&+ \sum_{k,m=1}^{n}\sum_{t,s=0}^{\gamma}\sum_{\nu=0}^{\tilde{n}_s}\sum_{\mu=0}^{n_s - 1} \Phi_k^{(\nu)}(u;\lambda_s)\ \overline{\Phi_m^{(\mu)}(v;\lambda_t)}\ \left[f_{k,s}^{(\nu)}, f_{m,t}^{(\mu)}\right].
\end{aligned} \tag{4.2.14}$$

The right hand side of (4.2.13) becomes

$$\begin{aligned}
&\sum_{i,j=1}^{n} \int_{-\infty}^{\infty} \Phi_i(\tilde{u};\lambda)\ \overline{\Phi_j(\tilde{v};\lambda)}\ \frac{d\sigma_{ij}(\lambda)}{\lambda |p(\lambda)|^2} = \\
&= \sum_{i,j=1}^{n} \int_{-\infty}^{\infty} \left(\Phi_i(u;\lambda) - \sum_{k=0}^{n}\sum_{s=0}^{\gamma}\sum_{\nu=0}^{\tilde{n}_s} \Phi_k^{(\mu)}(u;\lambda_s)\Phi_i(f_{k,s}^{(\nu)};\lambda)\right).
\end{aligned}$$

$$\cdot \left(\Phi_i(v;\lambda) - \sum_{k=0}^{n} \sum_{s=0}^{\gamma} \sum_{\nu=0}^{n_s-1} \Phi_k^{(\mu)}(v;\lambda_s)\, \Phi_j(f_{k,s}^{(\nu)};\lambda)\, \frac{d\sigma_{ij}(\lambda)}{\lambda |p(\lambda)|^2}\right). \qquad (4.2.15)$$

Now the statement of the theorem follows easily from (4.2.14) and (4.2.15).

4.3. Krein transformation corresponding to the operators A'_α.

In the sequel we suppose that the basic assumptions I - IV of Section 2.2 are satisfied. In order to apply Theorem 4.2.1, we take $H = L'_r$ and $T = A'_\alpha$ (see Section 2.6 and 3.1).

Next, we shall show that the system of linear functionals given by (3.4.5 - 6) is a Krein transformation corresponding to A'_α. That is, the mapping $f \mapsto \Phi(f;\lambda) = (\Phi_0(f;\lambda),\ldots, \Phi_{n-1}(f;\lambda))^T$ satisfies the conditions (i - iii) of Section 4.1. Recall that $\Phi(f;\lambda)$, $\Phi_0(f;\lambda),\ldots,\Phi_{n-1}(f;\lambda)$ depend all on α.

Indeed, the condition (i) is obviously satisfied. We check the condition (ii), i.e. that for a λ_1 the linear functionals $\Phi_0(\cdot;\lambda_1),\ldots, \Phi_{n-1}(\cdot;\lambda_1)$ are linearly independent on L'_r. Suppose to the contrary, that for some constants $c_0,\ldots,c_{n-1}$ we have

$$\sum_{j=0}^{n-1} c_j \Phi_j(f;\lambda_1) = 0 \quad \text{for all} \quad f \in L'_r. \qquad (4.3.1)$$

From the definition of $\Phi_j(f;\lambda_1)$, (4.3.1) is equivalent to

$$0 = \sum_{j=0}^{n-1} c_j \int_0^b r(t) f(t) \varphi_j(t;\lambda_1)\, dt = \int_0^b r(t) f(t) \sum_{j=0}^{n-1} c_j \varphi_j(t;\lambda_1)\, dt \quad \text{for all} \quad f \in L'_r. \qquad (4.3.2)$$

The relation (4.3.2) implies that

$$u(t) \equiv \sum_{j=0}^{n-1} c_j \varphi_j(t;\lambda_1) = 0 \quad \text{if} \quad t \text{ is in the support of } r, \qquad (4.3.3)$$

hence

$$\int_0^b r(t) |u(t)|^2 dt = 0.$$

But the function u satisfies $u^{[2n]} = \lambda r u$, and the assumption IV with Remark 2.2 imply that

$$\sum_{j=0}^{n-1} c_j \varphi_j(t;\lambda_1) = 0 \quad \text{for all} \quad t\,,$$

a contradiction to the fact that $\varphi_0,\dots,\varphi_{n-1}$ are linearly independent.

To see that also the condition (iii) is satisfied, consider $f \in L'_r$. Assume e.g., that $f(t) = 0$ for $t \geq b'$ with some $b' \in (0,b)$ and suppose

$$\mathfrak{D}_j(f;\lambda) = \int_0^b r(t)f(t)\varphi_j(t;\lambda_1)dt = 0\,, \quad j = 0,1,\dots,n-1\,.$$

For $0 \leq t \leq b' < b$, let $y(t)$ be a solution of the equation

$$y^{[2n]}(t) - \lambda r(t)y(t) = r(t)f(t)\,, \quad 0 \leq t \leq b' < b\,,$$

which satisfies the conditions

$$y^{[\nu]}(b') = 0\,, \quad \nu = 0,1,\dots,2n-1\,. \tag{4.3.4}$$

By a well-known existence theorem (compare Section 2.4), such a solution exists and is uniquely determined.

We shall show that $y \in D(A'_\alpha)$. Indeed,

$$\begin{aligned} 0 = \mathfrak{D}_j(f;\lambda) &= \int_0^b r(t)f(t)\varphi_j(t;\lambda)dt = \\ &= \int_0^b (y^{[2n]}(t) - \lambda r(t)y(t))\varphi_j(t;\lambda))dt = \\ &= \int_0^b y^{[2n]}(t)\varphi_j(t;\lambda)dt - \lambda\int_0^b r(t)y(t)\varphi_j(t;\lambda)dt\,. \end{aligned} \tag{4.3.5}$$

Using Lagrange's identity (see Section 2.1) in the first term on the right hand side of (4.3.5), we get

$$\begin{aligned} &\int_0^b y^{[2n]}(t)\varphi_j(t;\lambda)dt = \\ &= \int_0^b y(t)\varphi_j^{[2n]}(t;\lambda)dt + \sum_{\nu=1}^{n}\left[y^{[2n-\nu]}(t)\,\overline{\varphi_j^{[\nu-1]}} - y^{[\nu-1]}(t)\,\overline{\varphi_j^{[2n-\nu]}}\right]_0^{b'}. \end{aligned}$$

From (4.3.4), the relation (4.3.5) becomes

$$0 = \int_0^b y(t)(\varphi_j^{[2n]}(t;\lambda) - \lambda r(t)\varphi_j(t;\lambda))dt + \langle K\hat{y}, \hat{\varphi}_j\rangle = \langle K\hat{y}, \hat{\varphi}_j\rangle\,, \tag{4.3.6}$$

where $\hat{\varphi}_j$ is j:th column of the matrix

$$\varphi(0;\lambda) = \begin{pmatrix} \varphi(0,\lambda;1) \\ \varphi(0,\lambda;2) \end{pmatrix} = \begin{pmatrix} (\sin\alpha) \\ \tilde{I}_n(\cos\alpha) \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} 0 & -\tilde{I}_n \\ \tilde{I}_n & 0 \end{pmatrix}\,.$$

That is

$$\langle \hat{y}_1(0), (\cos\alpha)_j \rangle - \langle \tilde{I}_n \hat{y}_2(0), (\sin\alpha)_j \rangle = 0, \quad j = 0,1,\ldots,n-1 ,$$

and we get finally

$$(\cos\alpha)\, \hat{y}_1(0) - (\sin\alpha)\, \tilde{I}_n \hat{y}_2(0) = 0 .$$

Extending the function y by putting $y(t) = 0$ if $b' \le t < b$, it follows that $y \in D(A'_\alpha)$. Hence y belongs to L'_r and is a solution of the equation

$$(A'_\alpha - \lambda I)y = f .$$

On the other hand, if y is a solution of (4.3.7), then (4.3.6) holds and this can easily be transformed into (4.3.5), that is we get

$\Phi_j(f;\lambda) = 0$, $j = 0,1,\ldots,n-1$.

From Section 3.3 it follows (see (3.3.6)), that there exists a polynomial q of degree $\kappa_\alpha \le \gamma_\alpha$, such that

$$\{u,u\} = \left[A'_\alpha q(A'_\alpha)u,\ q(A'_\alpha)u\right] \ge 0 \qquad (u,v \in D((A'_\alpha)^{\kappa_\alpha+1})) .$$

Moreover, from Lemma 3.2.4 and the isomorphism between the spaces L'_r and L'_R and operators A'_α and T_α it follows that, for arbitrary $u \in D((A'_\alpha)^{\kappa_\alpha+1})$, there exists a sequence $(u_n) \subset D((A'_\alpha)^{\kappa_\alpha+2})$, such that

$$\left[A'_\alpha q(A'_\alpha)(u-u_n),\ q(A'_\alpha)(u-u_n)\right] \to 0 \qquad (n \to \infty) .$$

Summarizing these results, we see that the conditions 1) - 3) of Theorem 4.2.1 are satisfied for $T = A'_\alpha$. This gives the following result:

THEOREM 4.3.1. Suppose the assumptions I - IV are satisfied. Then there exists a polynomial q of degree $\kappa_\alpha \le \gamma_\alpha$:

$$q(\lambda) = \prod_{s=0}^{\gamma} (\lambda - \lambda_s)^{n_s}, \quad \lambda_0 = 0,\ n_0 \ge 0,\ n_s > 0 \quad (s = 1,2,\ldots,\gamma)$$

such that

$$\left[A'_\alpha\, q(A'_\alpha)\, u,\ q(A'_\alpha)\, u\right] \ge 0 \qquad (u \in D((A'_\alpha)^{\kappa_\alpha+1})) .$$

Moreover, there exists a function $\sigma \in S_n$ such that for $u \in L'_r$, $v \in D(A'_\alpha)$ we have

$$[u,v] = \sum_{i,j=1}^{n-1} \int_{-\infty}^{\infty} (\Phi_i(u;\lambda) - \sum_{k=0}^{n-1} \sum_{s=0}^{\gamma} \sum_{\nu=0}^{\tilde{n}_s} \Phi_k^{(\nu)}(u;\lambda_s)\Phi_i(f_{k,s}^{(\nu)};\lambda)) \cdot$$

$$\cdot \overline{(\Phi_j(v;\lambda) - \sum_{k=0}^{n-1} \sum_{s=0}^{\gamma} \sum_{\nu=0}^{\tilde{n}_s} \Phi_k^{(\nu)}(v;\lambda_s)\Phi_i(f_{k,s}^{(\nu)};\lambda))} \frac{d\sigma_{ij}(\lambda)}{\lambda|q(\lambda)|^2} +$$

$$+ \sum_{k=0}^{n-1} \sum_{s=0}^{\gamma} \sum_{\nu=0}^{\tilde{n}_s} \overline{\Phi_k^{(\nu)}(u;\lambda_s)} \left[f_{k,s}^{\prime\nu)}, v\right] +$$

$$+ \sum_{k=0}^{n-1} \sum_{s=0}^{\gamma} \sum_{\nu=0}^{n_s-1} \overline{\Phi_k^{(\nu)}(v;\lambda_s)} \left[u, f_{k,s}^{(\nu)}\right] -$$

$$- \sum_{k,m=0}^{n-1} \sum_{s,t=0}^{\gamma} \sum_{\nu=0}^{\tilde{n}_s} \sum_{\mu=0}^{n_s-1} \Phi_k^{(\nu)}(u;\lambda_s) \overline{\Phi_m^{(\mu)}(v;\lambda_t)} .$$

Here $f_{k,s}^{(\nu)} \in D(A'_\alpha)$ are elements with the properties (4.2.1) and $\tilde{n}_s = n_s - 1$ if $s > 0$, $\tilde{n}_0 = n_0$. If $u,v \in D((A'_\alpha)^{\kappa+1})$ the relation

$$\left[A'_\alpha\, q(A'_\alpha)\, u,\, q(A'_\alpha)\, v\right] = \sum_{i,j=0}^{n-1} \int_{-\infty}^{\infty} \Phi_i(u;\lambda)\, \overline{\Phi_j(v;\lambda)}\, d\sigma_{ij}(\lambda)$$

holds.

We mention, that the polynomial q is in general not uniquely determined.

5. EXPANSION THEOREMS FOR THE SECOND-ORDER CASE

5.1. Extension of the transformation Φ.

In the sequel we consider the second-order case:

$$-(\tfrac{1}{p}\, y')' + qy = rf \quad \text{on} \quad [0,b), \tag{5.1.1}$$

$$\cos\alpha y(0) - \sin\alpha y^{[1]}(0) = 0$$

for some α, $0 \le \alpha < \pi$.

We suppose again that the assumptions I, II′, III, IV′ from Section 2.9 are satisfied. Moreover, the equation (5.1.1) is supposed to be in r-limit (1,1) case at b, that is the operators A_α defined in Section 3.1 are J-selfadjoint.

With definitisable J-selfadjoint operator A_α, $\alpha \in [0,\pi)$ we have associated the "coefficient of Weyl" w_α (see Section 3.4.1) and the transformation Φ on L_r' (see Section 3.4), which in the following for $n=1$ will be denoted by Φ_α, that is:

$$\Phi_\alpha(f;\lambda) = \int_0^b f(x)\varphi_\alpha(x;\lambda)r(x)dx, \quad f \in L_r'$$

with φ_α from Section 3.4.1.

The transformation $f \mapsto \Phi_\alpha(f;\cdot)$ will now be extended to the whole space L_r^2 as follows. Suppose first that $f \in L_r'$. Then, from (3.4.23) for $n=1$ we have

$$\begin{aligned}\left[(A_\alpha - \lambda I)^{-1} f, f\right] &= \int_0^b g(x)\overline{f(x)}r(x)dx \\ &= w_\alpha(\lambda)\ \Phi_\alpha(f;\lambda)\ \Phi_\alpha(\overline{f};\lambda) + S(f,f;\lambda)\end{aligned}$$

and

$$\left[A_\alpha(A_\alpha - \lambda I)^{-1} f, f\right] = \lambda w_\alpha(\lambda)\ \Phi_\alpha(f;\lambda)\ \Phi_\alpha(\overline{f};\lambda) + \cdots$$

where $\cdots$ denotes an entire function in λ.

Moreover, if $\alpha \in (\alpha_0,\pi)$, where α_0 is given in Lemma 2.9.4, and the eigenvalue s_α is real: $s_\alpha = \overline{s_\alpha}$, then we have for $f \in D(A_\alpha^2)$

$$\left[A_\alpha(A_\alpha - s_\alpha I)(A_\alpha - s_\alpha I)^{-1} f,\ (A_\alpha - s_\alpha I)f\right] = \lambda(\lambda - s_\alpha)^2\ w_\alpha(\lambda)\Phi_\alpha(f;\lambda)\Phi_\alpha(\overline{f};\lambda) + \cdots .$$

Replacing $w_\alpha(\lambda)$ by $\hat{w}_\alpha(\lambda)$ (see (3.4.27)) and using Corollary 3.4.7, we find

$$\left[A_0(A_0-\lambda I)^{-1}f,f\right] = \lambda^2\,\hat{w}_0(\lambda)\Phi_0(f;\lambda)\Phi_0(\bar{f};\lambda) + \cdots$$

$$= \lambda^2\Big(c_0+c_1\lambda+\int_{-\infty}^{\infty}\Big(\frac{1}{t-\lambda}-\frac{t+\lambda}{1+t^2}\Big)(1+t^2)\,d\sigma_0(t)\Big)\cdot\Phi_0(f;\lambda)\Phi_0(\bar{f};\lambda) + \cdots,$$

$$\left[A_\alpha(A_\alpha-\lambda I)^{-1}f,f\right] = (\hat{w}_\alpha(\lambda)-\lambda\cot\alpha)\,\Phi_\alpha(f;\lambda)\Phi_\alpha(\bar{f};\lambda) + \cdots$$

$$= \Big(c_0+c_1\lambda+\int_{-\infty}^{\infty}\Big(\frac{1}{t-\lambda}-\frac{t+\lambda}{1+t^2}\Big)(1+t^2)\,d\sigma_\alpha(t)\Big)\cdot\Phi_\alpha(f;\lambda)\Phi_\alpha(\bar{f};\lambda) + \cdots.$$

Then, from the Stieltjes-Livšic inversion formula (see Appendix) we get for an arbitrary bounded but sufficiently large interval Δ

$$\left[A_0E_0(\Delta)f,f\right] = \int_\Delta t^2(1+t^2)\,|\,\Phi_0(f;t)\,|^2\,d\sigma_0(t), \tag{5.1.2}$$

$$\left[A_\alpha E_\alpha(\Delta)f,f\right] = \int_\Delta (1+t^2)\,|\,\Phi_\alpha(f;t)\,|^2\,d\sigma_\alpha(t), \tag{5.1.3}$$

if $\alpha\in(0,\alpha_0]$ or $\alpha\in(\alpha_0,\pi)$ and $s_\alpha\neq\bar{s}_\alpha$,

$$\left[A_\alpha E_\alpha(\Delta)(A_\alpha-s_\alpha I)f,\,(A_\alpha-s_\alpha I)f\right] = \int_\Delta (1+t^2)^2\,|\,\Phi_\alpha(f;t)\,|^2\,d\sigma_\alpha(t) + d_3\,|\,\Phi_\alpha(f;s_\alpha)\,|^2, \tag{5.1.4}$$

if $\alpha\in(\alpha_0,\pi)$, $s_\alpha=\bar{s}_\alpha$.

By Lemma 3.2.4 and Section 2.4, an arbitrary $f\in L_r'$ can be approximated in norm by a sequence of elements $f_n\in D(A_\alpha^2)$, whose support is contained in a fixed compact set of $[0,b)$. Therefore, the relations (5.1.2-4) remain true for all f with compact support.

Choose now $\lambda_0\in\rho(A_\alpha)$, consider $f\in D(A_\alpha^2)$ with compact support in $[0,b)$ and put $g=(A_\alpha-\lambda_0 I)^2 f$. Then, it is easy to see (compare Lemma 4.1.1) that

$$\Phi_\alpha(g;\lambda) = (\lambda-\lambda_0)^2\,\Phi_\alpha(f;\lambda),$$

and (5.1.2-4) yield the relations

$$\left[A_0E_0(\Delta)(A_0-\lambda_0 I)^{-2}g,\,(A_0-\lambda_0 I)^{-2}g\right] = \int_\Delta \frac{t^2(1+t^2)}{|t-\lambda_0|^4}\,|\,\Phi_0(g;t)\,|^2\,d\sigma_0(t); \tag{5.1.5}$$

$$\left[A_\alpha E_\alpha(\Delta)(A_\alpha - \lambda_0 I)^{-2} g, (A_\alpha - \lambda_0 I)^{-2} g\right] = \int_\Delta \frac{1+t^2}{|t-\lambda_0|^4} |\Phi_\alpha(g;t)|^2 d\sigma_\alpha(t), \tag{5.1.6}$$

if $\alpha \in (0,\alpha_0]$ or $\alpha \in (\alpha_0,\pi)$ and $s_\alpha \neq \overline{s_\alpha}$;

$$\left[A_\alpha(A_\alpha - s_\alpha I) E_\alpha(\Delta)(A_\alpha - \lambda_0 I)^{-2} g, (A_\alpha - s_\alpha I)(A_\alpha - \lambda_0 I)^{-2} g\right] =$$

$$= \int_\Delta \frac{(1+t^2)^2}{|t-\lambda_0|^4} |\Phi_\alpha(g;t)|^2 \, d\sigma_\alpha(t) + \frac{d_3}{|s_\alpha - \lambda_0|^4} |\Phi_\alpha(g;s_\alpha)|^2, \tag{5.1.7}$$

if $\alpha \in (\alpha_0,\pi)$, $s_\alpha = \overline{s_\alpha}$.

Now instead of Δ take a sequence (Δ_n), $\Delta_n \uparrow \mathbb{R}$. Then Lemma 1.5.1 implies that $E_\alpha(\Delta_n)(A_\alpha - \lambda_0 I)^{-1} g$ tends to $E_\alpha(\mathbb{R})(A_\alpha - \lambda_0 I)^{-1} g$ in norm, and we get for arbitrary $\alpha \in [0,\pi)$, that there exists a constant c_α such that

$$\int_{\mathbb{R}} |\Phi_\alpha(g;t)|^2 \, d\tilde{\sigma}_\alpha(t) \leq c_\alpha \|g\|^2, \tag{5.1.8}$$

where $\tilde{\sigma}_\alpha = \sigma_\alpha$ if $\alpha \in [0,\alpha_0]$ or $\alpha \in (\alpha_0,\pi)$, $s_\alpha \neq \overline{s_\alpha}$; otherwise the difference $\tilde{\sigma}_\alpha - \sigma_\alpha$ is the point measure $\frac{d_3}{|s_\alpha - \lambda_0|^4}$ at s_α.

The following lemma gives the possibility to extend the transformation Φ by continuity to the whole space L^2_r.

LEMMA 5.1.1. The set $M = \{(A_\alpha - \lambda_0 I)^2 h : h \in D(A_\alpha^2)$ with compact support in $[0,b)\}$ is dense in L^2_r.

Proof. From the representation of the resolvent (see Theorem 3.4.2), $f \in M$ if and only if it has compact support (suppose e.g. $f(x) = 0$ if $x \geq b'$, $b' \in (0,b)$),

$$\int_0^x \varphi_\alpha(t;\lambda_0) f(t) r(t) dt = 0 \quad \text{if} \quad x \geq b'$$

and

$$\int_0^x \varphi_\alpha(t;\lambda_0) \{\varphi_\alpha(t;\lambda_0) \int_t^b f(s)\psi_\alpha(s;\lambda_0) r(s) ds +$$

$$+ \psi_\alpha(t;\lambda_0) \int_0^t f(s) \varphi_\alpha(s;\lambda_0) r(s) ds\} r(t) dt = 0 \quad \text{if} \quad x \geq b'. \tag{5.1.9}$$

Consider on the dense set of all $f \in L'_r$ the linear functional

$$F : F(f) = \int_0^b \varphi_\alpha(x;\lambda_0) f(x) r(x) dx .$$

As the equation $-(\frac{1}{p}y')' + qy = rf$ is in r-limit (1,1) case at b and $\lambda_0 \in \rho(A_\alpha)$, we have $\varphi_\alpha(\cdot,\lambda_0) \notin L^2_r$, that is F is discontinuous. Therefore, by a consequence of Hahn-Banach's theorem, the set Z of its zeros is dense in L^2_r.

Define a linear functional F_1 on Z by the left hand side of (5.1.9). The term in square brackets is $(A_\alpha - \lambda_0 I)^{-1} g$, $g \in Z$, and the set of these elements is evidently dense in L^2_r. Therefore F_1 is also discontinuous and the lemma follows.

For arbitrary $g_0 \in L^2_r$, choose a sequence of elements $g_n \in M$, $n = 1,2,\ldots$, which converges to g_0. The inequality (5.1.8) implies the convergence of the sequence $(\Phi_\alpha(g_n;\cdot)$ in $L^2(\tilde{\sigma}_\alpha)$. Denote its limit for a moment by $\tilde{\Phi}_\alpha(g;\cdot)$. We shall show, that for $g_0 \in L^2_r$ with compact support the equality

$$\Phi_\alpha(g_0;\cdot) = \tilde{\Phi}_\alpha(g_0;\cdot) \quad \text{in} \quad L^2(\tilde{\sigma}_\alpha) \tag{5.1.10}$$

holds. Indeed, $g_n \to g_0$ implies $E_\alpha(\Delta) g_n \to E_\alpha(\Delta) g_0$ for each bounded, sufficiently large interval Δ. Therefore from (5.1.3 - 7)

$$\Phi_\alpha(g_n;\cdot) \to \Phi_\alpha(g_0;\cdot) \qquad \tilde{\sigma}_\alpha \text{ - a.e. on } \Delta,$$

that is, we have $\Phi_\alpha(g_0;\lambda) = \tilde{\Phi}_\alpha(g_0;\lambda)$ for $\tilde{\sigma}_\alpha$ - almost all $\lambda \in \Delta$. As Δ is arbitrarily large, relation (5.1.10) follows.

Now, for arbitrary $g \in L^2_r$, instead of $\tilde{\Phi}_\alpha(g_0;\cdot)$ we shall write $\Phi_\alpha(g;\cdot)$ and call this element of $L^2(\tilde{\sigma}_\alpha)$ the transformation Φ_α of g. Let us repeat that for $g \in L'_r$ as a representative in $L^2(\tilde{\sigma}_\alpha)$ the entire function

$$\Phi_\alpha(g;\cdot) = \int_0^b g(x) \varphi_\alpha(x;\lambda) r(x) dx$$

can be chosen.

5.2. The spectral measure. Expansion at ∞.

In the following, for given $\alpha \in [0,\pi)$ we fix an open interval $\Delta_0 : 0 \in \Delta_0$, an open set Δ_∞ being the complement of a compact interval and, if $\alpha \in (\alpha_0,\pi)$ and $s_\alpha = \overline{s_\alpha}$, an open interval $\Delta_\alpha : s_\alpha \in \Delta_\alpha$, such that Δ_0, Δ_∞ and Δ_α are mutually disjoint. By χ_0, χ_∞, χ_α we denote the characteristic function of $\Delta_0, \Delta_\infty, \Delta_\alpha$ resp. Let Δ be a bounded interval whose closure does not contain 0 and s_α.

With the operator A_α we have associated a non-decreasing bounded function σ_α by Corollary 3.4.7. Now we introduce the following σ-finite measure τ_α on $\mathbb{R}\setminus\{0, s_\alpha\}$, called the spectral measure of A_α:

$$d\tau_\alpha(t) = \begin{cases} t(1+t^2)\,d\sigma_0(t) & \text{if } \alpha = 0, \\ \dfrac{1+t^2}{t}\,d\sigma_\alpha(t) & \text{if } \alpha \in (0,\pi) \text{ and } s_\alpha \text{ is not a singular critical point of } A_\alpha, \\ \dfrac{(1+t^2)^2}{t(t-s_\alpha)^2}\,d\sigma_\alpha(t) & \text{if } \alpha \in (0,\pi),\ s_\alpha = \overline{s_\alpha} \text{ and } s_\alpha \text{ is a singular critical point of } A_\alpha. \end{cases}$$

The points 0, ∞ and s_α if $s_\alpha = \overline{s_\alpha}$ will be called critical points of τ_α. Let s be one of these critical points. If $\tau_\alpha(\Delta)$ $(= \int_\Delta d\tau_\alpha(t))$ is uniformly bounded for all intervals Δ in a certain neighbourhood Δ_s of s, $s \notin \overline{\Delta}$, the measure τ_α can evidently be extended to all Borel sets in Δ_s putting $\tau_\alpha(\{s\}) = 0$. In this case s is called a regular critical point, otherwise s will be called a singular critical point of the spectral measure τ_α. It is easy to see, that s is a regular (singular) critical point of τ_α if and only if it is a regular (singular resp.) critical point of A_α. In any case

$$\int_{\Delta_0} t\,d\tau_\alpha(t) < \infty; \quad \int_{\Delta_\infty} \frac{d\tau_\alpha(t)}{t} < \infty \quad \text{if } \alpha \in (0,\pi),$$

$$\int_{\Delta_\infty} \frac{d\tau_0(t)}{t^3} < \infty; \quad \int_{\Delta_\alpha} (t-s_\alpha)^2\,d\tau_\alpha(t) < \infty.$$

Here and in the following, if s is a singular critical point of τ_α, the integrals are to be understood as improper integrals with respect to

this singularity s.

Let now Δ be a bounded interval having a positive distance from 0 and s_α. Then (3.4.19), (3.4.25), Corollary 3.4.7 and the Stieltjes-Livšic inversion formula imply

$$[E_\alpha(\Delta)f,g] = \int_\Delta d\tau_\alpha(t)\Phi_\alpha(f;t)\int_0^b \varphi_\alpha(x;t)\overline{g(x)}r(x)dx \qquad (5.2.1)$$

for all $f,g \in L_r'$. An arbitrary $f \in L_r^2$ can be approximated in norm by a sequence $(f_n) \subset L_r'$. Then $\Phi_\alpha(f_n;\cdot)$ tends to $\Phi_\alpha(f;\cdot)$ in $L^2(\tilde{\sigma}_\alpha)$. Therefore (5.2.1) holds true for arbitrary $f \in L_r^2$ and we get

$$(E_\alpha(\Delta))f(x) = \int_\Delta \Phi_\alpha(f;t)\varphi_\alpha(x;t)d\tau_\alpha(t) \qquad (5.2.2)$$

where the integral on the right-hand side exists pointwise for all $x \in [0,b)$.

Suppose now $f \in D(A_\alpha)$. Then (5.2.2) and Lemma 1.5.1 imply

$$E_\alpha(\Delta_\infty)f = \lim_{\substack{n\to\infty \\ n'\to\infty}} E_\alpha(\Delta_\infty \cap [-n-n']) f = \lim_{\substack{n\to\infty \\ n'\to\infty}} \int_{\Delta_\infty\cap[-n,n']} \Phi_\alpha(f;t)\varphi_\alpha(\cdot;t)d\tau_\alpha(t),$$

where the limit exists in the norm of L_r^2, that is we have

$$\lim_{n\to\infty} \int_0^b \left| \int_{\Delta_\infty\cap[-n,n]} \Phi_\alpha(f;t)\varphi_\alpha(x;t)d\tau_\alpha(t) - (E_\alpha(\Delta_\infty)f)(x)\right|^2 |r(x)|dx = 0$$

5.3. Expansion at the real critical point s_α.

Let $\alpha \in (\alpha_0,\pi)$ and $s_\alpha = \bar{s}_\alpha$. Suppose first that s_α is a regular critical point, that is we have $\int_{\Delta_\alpha} d\tau_\alpha(t) < \infty$. Then the range $E_\alpha(\Delta_\alpha)$ decomposes (see (1.3.3)):

$$E_\alpha(\Delta_\alpha)L_r^2 = G_\alpha \dot{+} G_\alpha^{[\perp]},$$

where G_α denotes the algebraic eigenspace belonging to s_α and $G_\alpha^{[\perp]}$ is a Hilbert space with respect to the scalar product $[\cdot,\cdot]$ $(-[\cdot,\cdot])$ if $s_\alpha > 0$ ($s_\alpha < 0$ resp.). Moreover, we have the following representation of $E_\alpha(\Delta_\alpha)f$:

$$E_\alpha(\Delta_\alpha)f = \int_{\Delta_\alpha} E_\alpha(d\lambda)f + d_0(f)u_\alpha + d_1(f)\dot{u}_\alpha + d_2(f)\ddot{u}_\alpha \qquad (5.3.1)$$

(for definition of u_α, $\dot{u}_\alpha$, $\ddot{u}_\alpha$, see Corollary 3.3.7). Here d_0, d_1, d_2 are continuous linear functionals on L^2_r (also depending on α), and $d_1 \neq 0$ ($d_2 \neq 0$) implies $\dot{u}_\alpha \in L^2_r$ ($\ddot{u}_\alpha \in L^2_r$ resp.) so that (5.3.1) has sense in L^2_r. If $d_1 = d_2 = 0$, then

$$[u_\alpha, u_\alpha] \neq 0 \quad \text{and} \quad d_0(f) = \frac{[f, u_\alpha]}{[u_\alpha, u_\alpha]} .$$

Note that $\ddot{u}_\alpha \in L^2_r$ implies $\dot{u}_\alpha \in L^2_r$. Now (5.3.1) can be written in the form:

$$(E_\alpha(\Delta_\alpha)f)(x) = \int_{\Delta_\alpha} \Phi_\alpha(f;t)\varphi_\alpha(x;t)d\tau_\alpha(t) + d_0(f)u_\alpha + d_1(f)\dot{u}_\alpha + d_2(f)\ddot{u}_\alpha .$$

Here the integral converges with respect to the singularity at s_α in the sense of L^2_r and at the same time pointwise in $x \in [0,b)$.

Let now $s_\alpha = \bar{s}_\alpha$ be a singular critical point of A_α. We consider first functions $f,g \in L'_r$. Then $\Phi_\alpha(f;\cdot)$ and $\Phi_\alpha(g;\cdot)$ are entire functions, and (3.4.19), (3.4.27), Corollary 3.4.7 and the Stieltjes-Livšic inversion formula imply

$$\begin{aligned}[E_\alpha(\Delta_\alpha)f,g] = \int_{\Delta_\alpha} \{ & \Phi_\alpha(f;t)\Phi_\alpha(\bar{g};t) - \Phi_\alpha(f;s_\alpha)\Phi_\alpha(\bar{g};s_\alpha) - \\ & -(t-s_\alpha)\frac{\partial}{\partial\lambda}\left(\Phi_\alpha(f;\lambda)\Phi_\alpha(\bar{g};\lambda)\right)\Big|_{\lambda=s_\alpha} \} d\tau_\alpha(t) + \\ & + d_1\Phi_\alpha(f;s_\alpha)\Phi_\alpha(\bar{g};s_\alpha) + d_2\frac{\partial}{\partial\lambda}\left(\Phi_\alpha(f;\lambda)\Phi_\alpha(\bar{g};\lambda)\right)\Big|_{\lambda=s_\alpha} + \\ & + d_3\frac{\partial^2}{\partial\lambda^2}\left(\Phi_\alpha(f;\lambda)\Phi_\alpha(\bar{g};\lambda)\right)\Big|_{\lambda=s_\alpha} .\end{aligned} \tag{5.3.2}$$

Consider for fixed $x \in [0,b)$ the expression

$$\begin{aligned}\int_{\Delta_\alpha} \{ & \Phi_\alpha(f;t)\varphi_\alpha(x;t) - \Phi_\alpha(f;s_\alpha)\varphi_\alpha(x;s_\alpha) - (t-s_\alpha)\frac{\partial}{\partial\lambda}\left(\Phi_\alpha(f;\lambda)\varphi_\alpha(x;\lambda)\right)\Big|_{\lambda=s_\alpha} \} d\tau_\alpha(t) + \\ & + d_1\Phi_\alpha(f;s_\alpha)\varphi_\alpha(x;s_\alpha) + d_2\frac{\partial}{\partial\lambda}\left(\Phi_\alpha(f;\lambda)\varphi_\alpha(x;\lambda)\right)\Big|_{\lambda=s_\alpha} + \\ & + d_3\frac{\partial^2}{\partial\lambda^2}\left(\Phi_\alpha(f;\lambda)\varphi_\alpha(x;\lambda)\right)\Big|_{\lambda=s_\alpha} .\end{aligned} \tag{5.3.3}$$

Evidently, it defines a continuous function of x, and (5.3.2) implies that in L^2_r this function coincides with $E_\alpha(\Delta_\alpha)f$. But we know

from Corollary 3.3.7, that the last two terms in (5.3.3) do not necessarily belong to L^2_r. Therefore, the integral in (5.3.3) in general does not define an element of L^2_r although it exists pointwise. In the following we shall introduce a subspace $H_+ \subset L^2_r$, such that the integral in (5.3.3) converges in the corresponding negative norm.

We choose an open set U in the complex plane $\mathbb{C}$, $U \supset \Delta_\alpha \cup \Delta_0$, and a function ρ on $[0,b)$ with the properties:

$$|\varphi_\alpha(x;\lambda)| \le \rho(x), \quad |\dot\varphi_\alpha(x;\lambda)| \le \rho(x), \quad |\ddot\varphi(x;\lambda)| \le \rho(x) \quad (x \in [0,b), \lambda \in U)$$

($\dot\varphi_\alpha(x;\lambda) = \frac{\partial}{\partial\lambda}\varphi_\alpha(x;\lambda)$, $\ddot\varphi_\alpha(x;\lambda) = \frac{\partial^2}{\partial\lambda^2}\varphi_\alpha(x;\lambda)$). Let H_+ be the set of all $f \in L^2_r$ such that

$$\|f\|_+^2 = \int_0^b |f(x)|^2 \nu(x)|r(x)|dx < \infty, \quad \nu(x) = (1+\rho^2(x))(1+|r(x)|)(1+x^2).$$

The corresponding negative norm

$$\|g\|_- = \sup_{f \in H_+, f\neq 0} |(f,g)| \, \|f\|_+^{-1} \qquad (g \in L^2_r)$$

is given by

$$\|g\|_-^2 = \int_0^b |g(x)|^2 |r(x)| \nu(x)^{-1} dx .$$

The functions $\varphi_\alpha(\cdot;\lambda)$, $\dot\varphi_\alpha(\cdot;\lambda)$ and $\ddot\varphi_\alpha(\cdot;\lambda)$ for $\lambda \in U$ belong to H_-, as follows e.g. from

$$\int_0^b |\ddot\varphi_\alpha(x;\lambda)|^2 |r(x)| \nu(x)^{-1} dx \le \int_0^b (1+x^2)^{-1} dx < \infty .$$

If $f \in H_+$, $\lambda \in U$ we have

$$|\Phi_\alpha(f;\lambda)| \le \|f\|_+ \, \|\varphi_\alpha(\cdot,\lambda)\|_-$$

and corresponding relations for $\dot\Phi_\alpha(f;\lambda) = \frac{\partial}{\partial\lambda}\Phi_\alpha(f;\lambda)$ and $\ddot\Phi_\alpha(f;\lambda) = \frac{\partial^2}{\partial\lambda^2}\Phi_\alpha(f;\lambda)$.

It follows that $\Phi_\alpha(f;\lambda)$ is holomorphic in U. Moreover, if $f \in H_+$, the integral in (5.3.3), which we shall denote for a moment by $I(f)$, converges in the norm of H_-. Indeed, using Lagrange's form of the remainder term, we get

$$\|I(f)\|_-^2 = \int_0^b \Big| \int_{\Delta_\alpha} \int_{\lambda=s_\alpha}^{t} (t-\lambda) \frac{\partial^2(\Phi_\alpha(f;\mu)\varphi_\alpha(x;\mu))}{\partial\mu^2}\Big|_{\mu=\lambda} d\lambda\, d\tau_\alpha(t)\Big|^2 |r(x)| \cdot$$
$$\cdot \nu(x)^{-1} dx \le C\|f\|_+^2 ,$$

where the constant C depends on U and ρ. Therefore, if $f \in H_+$, we have

$$(E_\alpha(\Delta_\alpha)f)(x) = \int_{\Delta_\alpha} \{\Phi_\alpha(f;t)\varphi_\alpha(x;t) - \Phi_\alpha(f;s_\alpha)\varphi_\alpha(x;s_\alpha) -$$
$$- (t-s_\alpha)\frac{\partial}{\partial\lambda}(\Phi_\alpha(f;\lambda)\varphi_\alpha(x;\lambda))|_{\lambda=s_\alpha}\} d\tau_\alpha(t) + \qquad (5.3.4)$$
$$+ d_{01}(f)\varphi_\alpha(x;s_\alpha) + d_{02}(f)\dot\varphi_\alpha(x;s_\alpha) + d_{03}(f)\ddot\varphi_\alpha(x;s_\alpha) .$$

Here $d_{01}(f) = d_1\Phi_\alpha(f;s_\alpha) + d_2\dot\Phi_\alpha(f;s_\alpha) + d_3\ddot\Phi_\alpha(f;s_\alpha)$,
$d_{02}(f) = d_2\Phi_\alpha(f;s_\alpha) + 2d_3\dot\Phi_\alpha(f;s_\alpha)$ and $d_{03}(f) = d_3\Phi_\alpha(f;s_\alpha)$.

The integral in (5.3.4) exists pointwise in $x \in [0,b)$ and in the norm of H_-.

From (5.3.4) other representations of $E_\alpha(\Delta_\alpha)f$ can be derived:

1) Regularizing only the first factor of $\Phi_\alpha(f;\cdot)\varphi_\alpha(x;\cdot)$ we get for $f \in H_+$:

$$E_\alpha(\Delta_\alpha)f = \int_{\Delta_\alpha} \{\Phi_\alpha(f;t) - \Phi_\alpha(f;s_\alpha) - (t-s_\alpha)\dot\Phi_\alpha(f;s_\alpha)\}\varphi_\alpha(\cdot,t)d\tau_\alpha(t) +$$
$$+ \Phi_\alpha(f;s_\alpha)v_{11} + \dot\Phi_\alpha(f;s_\alpha)v_{12} + \ddot\Phi_\alpha(f;s_\alpha)v_{13} . \qquad (5.3.5)$$

Here v_{11}, v_{12}, v_{13} are elements of H_-; we have e.g.

$$v_{11} = d_1\varphi_\alpha(\cdot;s_\alpha) + d_2\dot\varphi_\alpha(\cdot;s_\alpha) + d_3\ddot\varphi_\alpha(\cdot;s_\alpha) +$$
$$+ \int_{\Delta_\alpha} (\varphi_\alpha(\cdot;t) - \varphi_\alpha(\cdot;s_\alpha) - (t-s_\alpha)\dot\varphi_\alpha(\cdot;s_\alpha))d\tau_\alpha(t) .$$

The integral in (5.3.5) exists again pointwise for $x \in [0,b)$ and in the norm of H_-.

2) Regularizing only the second factor of $\Phi_\alpha(f;\cdot)\varphi_\alpha(x;\cdot)$ we get for arbitrary $f \in L_r^2$

$$E_\alpha(\Delta_\alpha)f = \int_{\Delta_\alpha} \Phi_\alpha(f;t)\{\varphi_\alpha(\cdot;t) - \varphi_\alpha(\cdot;s_\alpha) - (t-s_\alpha)\dot\varphi_\alpha(\cdot;s_\alpha)\} d\tau_\alpha(t) +$$
$$+ d_{11}(f)\varphi_\alpha(\cdot;s_\alpha) + d_{12}(f)\dot\varphi_\alpha(\cdot;s_\alpha) + d_{13}(f)\ddot\varphi_\alpha(\cdot;s_\alpha) . \qquad (5.3.6)$$

Here d_{11}, d_{12}, d_{13} are the following continuous linear functionals on L_r^2:

$$d_{11}(f) = d_1\Phi_\alpha(f;s_\alpha) + d_2\dot{\Phi}_\alpha(f;s_\alpha) + d_3\ddot{\Phi}_\alpha(f;s_\alpha) +$$
$$+ \int_{\Delta_\alpha} \{\Phi_\alpha(f;t) - \Phi_\alpha(f;s_\alpha) - (t-s_\alpha)\dot{\Phi}_\alpha(f;s_\alpha)\}\, d\tau_\alpha(t),$$

$$d_{12}(f) = d_2\Phi_\alpha(f;s_\alpha) + 2d_3\Phi_\alpha(f;s_\alpha) + \int_{\Delta_\alpha} (t-s_\alpha)(\Phi_\alpha(f;t) - \Phi_\alpha(f;s_\alpha))d\tau_\alpha(t),$$

$$d_{13}(f) = d_3\Phi_\alpha(f;s_\alpha).$$

Indeed, if $f \in H_+$, relation (5.3.6) follows immediately from (5.3.4). Moreover, $f_n \to f$ in L_r^2 implies $\Phi_\alpha(f_n;\,) \to \Phi_\alpha(f;\cdot)$ $(n \to \infty)$ in $L^2(\tilde{\sigma}_\alpha)$. If we denote the integral in (5.3.6) by $\Phi(f)$, we have therefore $\tilde{I}(f_n) \to \tilde{I}(f)$ $(n \to \infty)$ pointwise in $x \in [0,b)$ and in H_-, and $\tilde{I}$ is continuous as a mapping from L_r^2 into H_-. Obviously, the same is true for $E_\alpha(\Delta_\alpha)$. Therefore, also

$$f \to d_{11}(f)\varphi_\alpha(\cdot;s_\alpha) + d_{12}(f)\dot{\varphi}_\alpha(\cdot;s_\alpha) + d_{13}(f)\ddot{\varphi}_\alpha(\cdot;s_\alpha)$$

is a continuous mapping from L_r^2 into H_-.

The continuity of d_{1j} $(j=1,2,3)$ in L_r^2 will follow, if we only show that $\varphi_\alpha(\cdot;s_\alpha)$, $\dot{\varphi}_\alpha(\cdot;s_\alpha)$ and $\ddot{\varphi}_\alpha(\cdot;s_\alpha)$ are linearly independent elements of H_-. To see this, consider with some constants c_0, c_1, c_2 the equation

$$c_0\varphi_\alpha(\cdot;s_\alpha) + c_1\dot{\varphi}_\alpha(\cdot;s_\alpha) + c_2\ddot{\varphi}_\alpha(\cdot;s_\alpha) = 0 \quad \text{in} \quad H_-.$$

Denoting the function on the left-hand side by y, the relations

$$-(p^{-1}\dot{\varphi}')' + q\dot{\varphi}_\alpha - \lambda r\dot{\varphi}_\alpha = r\varphi_\alpha,$$

$$-(p^{-1}\ddot{\varphi}')' + q\ddot{\varphi}_\alpha - \lambda r\ddot{\varphi}_\alpha = r\dot{\varphi}_\alpha$$

imply that y satisfies the conditions of Lemma 2.9.1 $(b' = 0)$. Hence $y(x) = y^{[1]}(x) = 0$. Now it is not difficult to see that $c_0 = c_1 = c_2 = 0$.

5.4. Expansion at 0.

If $\alpha \in (0,\pi)$, we get from (3.4.25), Corollary 3.4.27 and the definition of τ_α

$$w_\alpha(\lambda) = \int_{\Delta_0} \left(\frac{1}{t-\lambda} + \frac{1}{\lambda}\right) d\tau_\alpha(t) + \frac{c_1}{\lambda} + \frac{c_2}{\lambda^2} + \ldots \tag{5.4.1}$$

Here $\ldots$ denotes a function holomorphic in Δ_0. Relations (3.4.19), (5.4.1) and the Stieltjes-Livšic inversion formula imply for $f,g \in L^2_r$ with compact support

$$\left[E_\alpha(\Delta_0)f,g\right] = \int_{\Delta_0} \left(\Phi_\alpha(f;t)\Phi_\alpha(\bar{g};t) - \Phi_\alpha(f;0)\Phi_\alpha(\bar{g};0)\right) d\tau_\alpha(t) + c_1\Phi_\alpha(f;0)\Phi_\alpha(\bar{g};0) + c_2 \frac{\partial}{\partial\lambda}\left(\Phi_\alpha(f;\lambda)\Phi_\alpha(\bar{g};\lambda)\right)\Big|_{\lambda=0}.$$

Now the considerations leading from (5.3.2) to (5.3.4) can be repeated, and we get the following representations:

$$\left(E_\alpha(\Delta_0)f\right)(x) = \int_{\Delta_0} \left\{\Phi_\alpha(f;t)\varphi_\alpha(x;t) - \Phi_\alpha(f;0)\varphi_\alpha(x;0)\right\} d\tau_\alpha(t) + c_1\Phi_\alpha(f;0)\varphi_\alpha(x;0) + c_2 \frac{\partial}{\partial\lambda}\left(\Phi_\alpha(f;\lambda)\varphi_\alpha(x;\lambda)\right)\Big|_{\lambda=0}; \tag{5.4.2}$$

$$\left(E_\alpha(\Delta_0)f\right) = \int_{\Delta_0} \left\{\Phi_\alpha(f;t) - \Phi_\alpha(f;0)\right\} \varphi_\alpha(\cdot;t) d\tau_\alpha(t) + \Phi_\alpha(f;0)w_{11} + \dot{\Phi}_\alpha(f;0)w_{12}; \tag{5.4.3}$$

$$\left(E_\alpha(\Delta_0)f\right) = \int_{\Delta_0} \Phi_\alpha(f;t)\left\{\varphi_\alpha(\cdot;t) - \varphi_\alpha(\cdot;0)\right\} d\tau_\alpha(t) + c_{11}(f)\varphi_\alpha(\cdot;0) + c_{12}(f)\dot{\varphi}_\alpha(\cdot;0), \tag{5.4.4}$$

where (5.4.2) and (5.4.3) are valid if $f \in H_+$, (5.4.4) holds for $f \in L^2_r$. The integrals exist pointwise in $x \in [0,b)$ and in the norm of H_-, w_{11}, w_{12} are elements of H_- and the linear functionals c_{11}, c_{12} are continuous on L^2_r. We remind that the functions $\varphi_\alpha(\cdot;0)$, $\dot{\varphi}_\alpha(\cdot;0)$ can belong to L^2_r only if $\alpha = \alpha_0$, but they always belong to H_-.

The corresponding considerations in the case $\alpha = 0$ can be treated simularily.

5.5. The general expansion theorem.

In order to write the above representation in a more comprehensive way, with the intervals Δ_0, Δ_α fixed as above we introduce the following notations; here p and q are functions on $\mathbb{R}$ with a first derivative:

$$R_0^0(p;q)(t) = p(t)q(t) - p(0)q(0)\chi_0(t),$$

$$R_1^0(p;q)(t) = p(t)(q(t) - q(0)\chi_0(t)),$$

$$R_0^{0,s_\alpha}(p;q)(t) = p(t)q(t) - p(0)q(0)\chi_0(t) - \\ - [p(s_\alpha)q(s_\alpha) + (t - s_\alpha)(pq)'(s_\alpha)]\chi_\alpha(t),$$

$$R_1^{0,s_\alpha}(p;q)(t) = p(t)\{q(t) - q(0)\chi_0(t) - [q(s_\alpha) + (t - s_\alpha)q'(s_\alpha)]\chi_\alpha(t)\}.$$

For arbitrary $f \in L_r^2$ we have

$$f = E_\alpha(\Delta_0)f + E_\alpha(\Delta_\infty)f + E_\alpha(\mathbb{R} \setminus (\Delta_0 \cup \Delta_\infty))f \quad \text{if} \quad \alpha \in [0,\alpha_0],$$

$$f = E_\alpha(\Delta_0)f + E_\alpha(\Delta_\infty)f + E_\alpha(\Delta_\alpha f) + E_\alpha(\mathbb{R} \setminus (\Delta_0 \cup \Delta_\infty \cup \Delta_\alpha))f \quad \text{if} \quad \alpha \in (\alpha_0,\pi), \\ s_\alpha = \overline{s_\alpha},$$

$$f = E_\alpha(\Delta_0)f + E_\alpha(\Delta_\infty)f + E_\alpha(\mathbb{R} \setminus (\Delta_0 \cup \Delta_\infty))f + \\ + \frac{[f, \varphi_\alpha(\cdot;\overline{s}_\alpha)]}{[\varphi_\alpha(\cdot;s_\alpha), \varphi_\alpha(\cdot;\overline{s}_\alpha)]}\varphi_\alpha(\cdot;s_\alpha) + \frac{[f, \varphi_\alpha(\cdot;s_\alpha)]}{[\varphi_\alpha(\cdot;\overline{s}_\alpha), \varphi_\alpha(\cdot;s_\alpha)]}\varphi_\alpha(\cdot;\overline{s}_\alpha) \\ \text{if} \quad \alpha \in (\alpha,\pi), \ s_\alpha \neq \overline{s}_\alpha.$$

THEOREM 5.5.1. Suppose (5.1.1) is in r-limit (1,1) case at b and for fixed $\alpha \in [0,\pi)$ the intervals Δ_0, Δ_∞, Δ_α are chosen as above. Then, there exist continuous linear functionals $c_{01}^{(\alpha)},\dots,d_{03}^{(\alpha)}$ on H_-, $c_{11}^{(\alpha)},\dots,d_{13}^{(\alpha)}$ on L_r^2, such that with the spectral function τ_α of the operator A_α, the following expansions hold for $f \in H_+ \cap D(A_\alpha)$ if $j = 0$ and for $f \in L_r^2 \cap D(A_\alpha)$ if $j = 1$:

0) If $\alpha \in [0,\alpha_0]$ we have

$$f(x) = \int_{-\infty}^{\infty} R_j^0(\Phi_\alpha(f;\cdot);\varphi_\alpha(x;\cdot))(t)d\tau_\alpha(t) + c_{j1}^{(\alpha)}(f)\varphi_\alpha(x;0) + c_{j2}^{(\alpha)}(f)\dot{\varphi}_\alpha(x;0).$$

1) If $\alpha \in (\alpha_0,\pi)$ and $s_\alpha \neq \overline{s}_\alpha$, then $\varphi_\alpha(\cdot;s_\alpha)$ and $\varphi_\alpha(\cdot;\overline{s}_\alpha) = \overline{\varphi_\alpha(\cdot;s_\alpha)}$

belong to L^2_r and are eigenvectors of A_α to the eigenvalues s_α and $\bar{s}_\alpha$ resp. and we have

$$f(x) = \int_{-\infty}^{\infty} R_j^0(\Phi_\alpha(f;\cdot);\ \varphi_\alpha(x;\cdot))(t)d\tau_\alpha(t) + c_{j1}^{(\alpha)}(f)\varphi_\alpha(x;0) +$$
$$+ c_{j2}^{(\alpha)}(f)\dot{\varphi}_\alpha(x;0) + \frac{[f,\ \varphi_\alpha(\cdot;\bar{s}_\alpha)]}{[\varphi_\alpha(\cdot;s_\alpha),\ \varphi_\alpha(\cdot;\bar{s}_\alpha)]}\varphi_\alpha(x;s_\alpha) +$$
$$+ \frac{[f,\ \varphi_\alpha(\cdot;s_\alpha)]}{[\varphi_\alpha(\cdot;s_\alpha),\ \varphi_\alpha(\cdot;\bar{s}_\alpha)]}\varphi_\alpha(x;\bar{s}_\alpha)\ .$$

2) If $\alpha \in (\alpha_0,\pi)$ and $s_\alpha = \bar{s}_\alpha$, then $\varphi_\alpha(\cdot,s_\alpha)$ $(\in L^2_r)$ is an eigenvector of A_α to the eigenvalue s_α, and one and only one of the following three expansions formula holds:

2a) $[\varphi_\alpha(\cdot;s_\alpha),\ \varphi_\alpha(\cdot;s_\alpha)] \neq 0$, then

$$f(x) = \int_{-\infty}^{\infty} R_j^0(\Phi_\alpha(f;\cdot);\ \varphi_\alpha(x;\cdot))(t)d\tau_\alpha(t) + c_{j1}^{(\alpha)}(f)\varphi_\alpha(x;0) +$$
$$+ c_{j2}^{(\alpha)}(f)\dot{\varphi}_\alpha(x;0) + \frac{[f,\ \varphi_\alpha(\cdot,s_\alpha)]}{[\varphi_\alpha(\cdot;s_\alpha),\ \varphi_\alpha(\cdot;s_\alpha)]}\varphi_\alpha(x;s_\alpha)\ .$$

2b) $[\varphi_\alpha(\cdot;s_\alpha),\ \varphi_\alpha(\cdot;s_\alpha)] = 0$ and $\left|\int_{\Delta_\alpha} d\tau_\alpha(t)\right| < \infty$, then

$$f(x) = \int_{-\infty}^{\infty} R_j^0(\Phi_\alpha(f;\cdot);\ \varphi_\alpha(x;\cdot))(t)d\tau_\alpha(t) + c_{j1}^{(\alpha)}(f)\varphi_\alpha(x;0) +$$
$$+ c_{j2}^{(\alpha)}(f)\dot{\varphi}_\alpha(x;0) + d_{11}^{(\alpha)}(f)\varphi_\alpha(\cdot;s_\alpha) + d_{12}^{(\alpha)}(f)\dot{\varphi}_\alpha(\cdot;s_\alpha) +$$
$$+ d_{13}^{(\alpha)}(f)\ddot{\varphi}_\alpha(\cdot;s_\alpha)\ .$$

2c) $\left|\int_{\Delta_\alpha} d\tau_\alpha(t)\right| = \infty$, then

$$f(x) = \int_{-\infty}^{\infty} R_j^{0,s_\alpha}(\Phi_\alpha(f;\cdot);\ \varphi_\alpha(x;\cdot))(t)d\tau_\alpha(t) + c_{j1}^{(\alpha)}(f)\varphi_\alpha(x;0) +$$
$$+ c_{j2}^{(\alpha)}(f)\dot{\varphi}_\alpha(x;0) + d_{j1}^{(\alpha)}(f)\varphi_\alpha(x;s_\alpha) + d_{j2}^{(\alpha)}(f)\dot{\varphi}_\alpha(x;s_\alpha) +$$
$$+ d_{j3}^{(\alpha)}(f)\ddot{\varphi}_\alpha(x;s_\alpha)\ .$$

In these representations the integrals are to be considered as improper intervals with respect to 0, ∞ and s_α. They converge pointwise in

$x \in [0,b)$ and in the norm of H_-.

The functions $c_{01}^{(\alpha)},\ldots,d_{13}^{(\alpha)}$ of Theorem 5.2 are explicitly given in (5.3.4), (5.3.6), (5.4.2) and (5.4.4).

The cases $j=0$ and $j=1$ in Theorem 5.2 correspond to the regularization of $\Phi_\alpha(f;\cdot)\varphi_\alpha(x;\cdot)$ and the regularization of the second factor $\varphi_\alpha(x;\cdot)$ resp. It is also possible to combine the formula (5.3.5) and (5.4.3), in order to get expansions where only the first factor $\Phi_\alpha(f;\cdot)$ is being regularized. These expansions are valid if $f \in H_+ \cap D(A_\alpha)$.

APPENDIX

THE STIELTJES-LIVŠIC INVERSION FORMULA

(I.S. Kac and M.G. Krein [13])

Let G be a simply connected domain in the complex plane containing a bounded interval Δ of the real axis and let f and g be two functions holomorphic in G. For a given function σ of bounded variation on the real axis such that

$$\int_{-\infty}^{\infty} \frac{|d\sigma(t)|}{1+|t|}\, dt < \infty,$$

define

$$F(z) = f(z) + g(z) \int_{-\infty}^{\infty} \frac{1}{(t-z)}\, d\sigma(t) \quad (z \in G,\ z \neq \bar{z}).$$

Evidently, the integral $\int_{-\infty}^{\infty} \frac{d\sigma(t)}{t-z}$ converges absolutely.

Then with the contour $\Gamma(\Delta;\varepsilon)$ given in Section 1.3 contained in G, the following relation holds:

$$\lim_{\varepsilon \to 0} \left(-\frac{1}{2\pi i} \int_{\Gamma(\Delta;\varepsilon)} F(z)\, dz \right) = \int_{\Delta} g(t)\, d\tilde{\sigma}(t) \quad \left(\tilde{\sigma}(t) = \frac{\sigma(t+0) + \sigma(t-0)}{2}\right).$$

To prove this, choose an interval Δ' containing Δ in its interior and contained in G. Then

$$F(z) = f(z) + g(z) \int_{(-\infty,+\infty)\setminus\Delta'} \frac{1}{(t-z)}\, d\sigma(t) + \int_{\Delta'} \frac{g(z)-g(t)}{z-t}\, d\sigma(t) + \int_{\Delta'} \frac{g(t)}{(t-z)}\, d\sigma(t).$$

The first three members on the right hand side are holomorphic in the interval Δ', and from the Stieltjes inversion formula the statement follows.

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